Structured Error Recovery for Codeword-Stabilized Quantum Codes

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Codeword stabilized (CWS) codes are, in general, non-additive quantum codes that can correct errors by an exhaustive search of different error patterns, similar to the way that we decode classical non-linear codes. For an n-qubit quantum code correcting errors on up to t qubits, this brute-force approach consecutively tests different errors of weight t or less, and employs a separate n-qubit measurement in each test. In this paper, we suggest an error grouping technique that allows to simultaneously test large groups of errors in a single measurement. This structured error recovery technique exponentially reduces the number of measurements by about 3^t times. While it still leaves exponentially many measurements for a generic CWS code, the technique is equivalent to syndrome-based recovery for the special case of additive CWS codes.

I. INTRODUCTION

Quantum computation makes it possible to achieve polynomial complexity for many classical problems that are believed to be hard [1, 2]. To preserve coherence, quantum operations need to be protected by quantum error correcting codes (QECCs) [3–5]. With error probabilities in elementary gates below a certain threshold, one can use multiple layers of encoding (concatenation) to reduce errors at each level and ultimately make arbitrarily-long quantum computation possible [6–14].

The actual value of the threshold error probability strongly depends on the assumptions of the error model and on the chosen architecture, and presently varies from $10^{-3}\%$ for a chain of qubits with nearest-neighbor couplings [15] and 0.7% for qubits with nearest-neighbor couplings in two dimensions [14], to 3% with postselection [12], or even above 10% if additional constraints on errors are imposed [13].

The quoted estimates have been made using stabilizer codes, an important class of codes which originate from additive quaternary codes, and have a particularly simple structure based on Abelian groups [16, 17]. Recently, a more general class of codeword stabilized (CWS) quantum codes was introduced in Refs. [18–21]. This class includes stabilizer codes, but is more directly related to non-linear classical codes.

This direct relation to classical codes is, arguably, the most important advantage of the CWS framework. Specifically, the classical code associated with a given CWS quantum code has to correct certain error patterns induced by a graph associated with the code. The graph also determines the graph state [22] serving as a starting point for an encoding algorithm exploiting the structure of the classical code [20]. With the help of powerful techniques from the theory of classical codes, already several new families of non-additive codes have been discovered, including codes with parameters proven to be superior to any stabilizer code [18–20, 23–26].

Both classical additive codes and additive quantum codes can be corrected by first finding the syndrome of a corrupted vector or quantum state, respectively, and then looking up the corresponding error (coset leader) in a precomputed table [27]. This is not the case for non-linear codes. In fact, even the notions of a syndrome and a coset become invalid for general non-linear codes. Furthermore, since quantum error correction must preserve the original quantum state in all intermediate measurements, it is more restrictive than many classical algorithms. Therefore, the design of a useful CWS code must be complemented by an efficient quantum error correction algorithm.

The goal of this work is to address this important unresolved problem for binary CWS codes. First, we design a procedure to *detect* an error in a narrower class, the *union stabilizer* (USt) codes, which possess some partial group structure [23, 24, 28]. Then, for a general CWS code and a set of graph-induced maps of correctable errors forming a group, we construct an auxiliary USt code which is the union of the images of the original CWS code shifted by all the elements of the group. Finally, we construct Abelian groups associated with correctable errors located on certain *index sets* of qubits. The actual error is found by first applying error-detection to locate the index set with the relevant auxiliary USt code, then using a collection of smaller USt codes to pinpoint the error in the group. Since we process large groups of er-

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rors simultaneously, we make a significant reduction of the number of measurements compared with the brute force error correction for non-linear (quantum or classical) codes.

More precisely, we consider an arbitrary distance-d CWS code $((n, K, d))$ that uses n qubits to encode a Hilbert space of dimension K and can correct all t-qubit errors, where $t = \lfloor (d-1)/2 \rfloor$. In Sec. II we give a brief overview of the notations and relevant facts from the theory of quantum error correction. Then in Sec. III, we construct a reference recovery algorithm that deals with errors individually. This algorithm requires up to $B(n,t)$ measurements, where

$$B(n,t) \equiv \sum_{i=0}^{t} \binom{n}{i} 3^i \tag{1}$$

is the total number of errors of size up to t (this bound is tight for non-degenerate codes). Each of these measurements requires up to $n^2 + K\mathcal{O}(n)$ two-qubit gates. In order to eventually reduce the overall complexity, we consider the special case of USt codes in Sec. IV. Here we design an error-detecting measurement for a USt code with a translation set of size K that requires $\mathcal{O}(Kn^2)$ two-qubit gates to identify a single error. Our error grouping technique presented in Sec. V utilizes such a measurement to check for several errors at once. For additive CWS codes the technique reduces to stabilizer-based recovery [Sec. V C]. In the case of generic CWS codes [Sec. V D], we can simultaneously check for all errors located on size-t qubit clusters; graph-induced maps of these errors form groups of size up to 2^{2t}. Searching for errors in blocks of this size requires up to $\binom{n}{t} - 1$ measurements to locate the cluster, plus up to $2t$ additional measurements to locate the error inside the group. In Sec. VI we discuss the obtained results and outline the directions of further study. Finally, in Appendix A we consider some details of the structure of corrupted spaces for the codes discussed in this work.

Note that some of the reported results have been previously announced in Ref. [29].

II. BACKGROUND

A. Notations

Throughout the paper, $\mathcal{H}_2 = \mathbb{C}^2$ denotes the complex Hilbert space that consists of all possible states $\alpha|0\rangle + \beta|1\rangle$ of a single qubit, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Correspondingly, we use the space $\mathcal{H}_2^{\otimes n} = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n}$ to represent any n-qubit state. Also,

$$\mathscr{P}_n \equiv i^m \{I, X, Y, Z\}^{\otimes n}, \quad m = 0, \ldots, 3 \tag{2}$$

denotes the Pauli group of size 2^{2n+2}, where X, Y, Z are the usual (Hermitian) Pauli matrices and I is the identity matrix. The members of this group are called Pauli operators; the operators in (2) with $m = 0$ form a basis of the vector space that consists of all operators acting on n-qubit states. The *weight* $\mathrm{wgt}(E)$ of a Pauli operator E is the number of terms in the tensor product (2) which are not a scalar multiple of identity. Up to an overall phase, a Pauli operator can be specified in terms of two binary strings, $\mathbf{v}$ and $\mathbf{u}$,

$$U = Z^{\mathbf{v}} X^{\mathbf{u}} \equiv Z_1^{v_1} Z_2^{v_2} \ldots Z_n^{v_n} X_1^{u_1} X_2^{u_2} \ldots X_n^{u_n}.$$

Hermitian operators in $\mathscr{P}_n$ have eigenvalues equal to 1 or -1. Generally, unitary operators (which can be outside of the Pauli group) which are also Hermitian, i.e., all eigenvalues are ± 1, will be particularly important in the discussion of measurements. We will call these operators *measurement operators*. Indeed, for such an operator M, a measurement gives a Boolean outcome and can be constructed with the help of a single ancilla, two Hadamard gates, and a controlled M gate [16] (see Fig. 1). The algebra of measurement operators is related to the algebra of projection operators discussed in [30], but the former operators, being unitary, are more convenient in circuits.

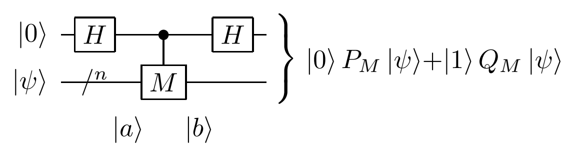

FIG. 1: Measurement of an observable M with all eigenvalues ± 1. The first Hadamard gate prepares the ancilla in the state $(|0\rangle + |1\rangle) |\psi\rangle / \sqrt{2}$, hence $|a\rangle = (|0\rangle + |1\rangle) |\psi\rangle / \sqrt{2}$. The controlled-$M$ gate returns $|b\rangle = C^M |a\rangle = (|0\rangle |\psi\rangle + |1\rangle M |\psi\rangle)/\sqrt{2}$. The second Hadamard gate finishes the incomplete measurement, $|c\rangle = |0\rangle P_M |\psi\rangle + |1\rangle Q_M |\psi\rangle$, where we used the projector identities (3). If the outcome of the ancilla measurement is $|0\rangle$, the result is the projection of the initial n-qubit state $|\psi\rangle$ onto the $+1$ eigenspace of M ($P_M |\psi\rangle$), otherwise it is the projection onto the -1 eigenspace of M ($Q_M |\psi\rangle$). For an input state $|1\rangle |\psi\rangle$ with ancilla in the state $|1\rangle$, the circuit returns $|1\rangle P_M |\psi\rangle + |0\rangle Q_M |\psi\rangle$.

A measurement of an observable defined by a Pauli operator M will be also called Pauli measurement [31]. For lack of a better term, other measurements will be called *non-Pauli*; typically the corresponding circuits are much more complicated than those for Pauli measurements.

We say that a state $|\psi\rangle \in \mathcal{H}_2^{\otimes n}$ is stabilized (anti-stabilized) by a measurement operator M if $M|\psi\rangle = |\psi\rangle$ ($M|\psi\rangle = -|\psi\rangle$). The corresponding projectors onto the positive and negative eigenspace are denoted by P_M and Q_M, respectively; they satisfy the identities

$$M = P_M - Q_M = 2P_M - \mathbb{1} = \mathbb{1} - 2Q_M. \tag{3}$$

We say that a space $\mathcal{Q}$ is stabilized by a set of operators $\mathcal{M}$ if each vector in $\mathcal{Q}$ is stabilized by each operator in $\mathcal{M}$. We use $\mathcal{P}(\mathcal{M})$ to denote the maximum space stabilized by $\mathcal{M}$, and $\mathcal{P}^{\perp}(\mathcal{M})$ to denote the corresponding orthogonal complement. For a set $\mathcal{M}$ of measurement operators,

each state in $\mathcal{P}^{\perp}(\mathcal{M})$ is anti-stabilized by some operator in $\mathcal{M}$.

When discussing complexity, we will quote the two-qubit complexity which just counts the total number of two-qubit gates. Thus, we ignore any communication overhead, as well as any overhead associated with single-qubit gates. For example, the complexity of the measurement in Fig. 1 is just that of the controlled-M gate operating on $n+1$ qubits [39]. For all circuits we discuss, the total number of gates (single- and two-qubit) is of the same order in n as the two-qubit complexity.

B. General QECCs

A general n-qubit quantum code $\mathcal{Q}$ encoding K quantum states is a K-dimensional subspace of the Hilbert space $\mathcal{H}_2^{\otimes n}$. Let $\{|i\rangle\}_{i=1}^{K}$ be an orthonormal basis of the K-dimensional code $\mathcal{Q}$ and let $\mathcal{E} \subset \mathscr{P}_n$ be some set of Pauli errors. The overall phase of an error [i^m in Eq. (2)] is irrelevant and will be largely ignored. The code detects all errors $E \in \mathcal{E}$ if and only if [2, 16]

$$\langle j|E|i\rangle = C_E \delta_{ij} \tag{4}$$

where C_E only depends on the error E, but is independent of the basis vectors [32]. The code has distance d if it can detect all Pauli errors of weight $(d-1)$, but not all errors of weight d. Such a code is denoted by $((n, K, d))$.

The necessary and sufficient condition for correcting errors in $\mathcal{E}$ is that all non-trivial combinations of errors from $\mathcal{E}$ are detectable. This gives [4, 5]

$$\langle j|E_1^{\dagger} E_2|i\rangle = C_{E_1,E_2} \delta_{ij}, \tag{5}$$

where $E_1, E_2 \in \mathcal{E}$ and, again, C_{E_1,E_2} is the same for all basis states i, j. A distance-d code corrects all errors of weight s such that $2s \leq d-1$, that is, $s \leq t \equiv \lfloor (d-1)/2 \rfloor$.

The code is *non-degenerate* if linearly independent errors from $\mathcal{E}$ produce corrupted spaces $E(\mathcal{Q}) \equiv \{E\,|\psi\rangle : |\psi\rangle \in \mathcal{Q}\}$ whose intersection is trivial (equals to $\{0\}$); otherwise the code is *degenerate* [17]. A stricter condition that the code is *pure* (with respect to $\mathcal{E}$) requires that the corrupted spaces $E_1(\mathcal{Q})$ and $E_2(\mathcal{Q})$ be mutually orthogonal for all linearly independent correctable errors E_1, $E_2 \in \mathcal{E}$.

For a degenerate code, we call a pair of correctable errors $E_1, E_2 \in \mathcal{E}$ *mutually-degenerate* if the corrupted spaces $E_1(\mathcal{Q})$ and $E_2(\mathcal{Q})$ coincide. Such errors belong to the same *degeneracy class*. For recovery, one only needs to identify the degeneracy class of the error that happened. The operators like $E_1^{\dagger} E_2$, connecting mutually-degenerate correctable errors E_1 and E_2, have no effect on the code and can be ignored.

As shown in Appendix A, for all codes discussed in this work, any two correctable errors E_1, E_2 yield corrupted spaces $E_1(\mathcal{Q})$, $E_2(\mathcal{Q})$ that are either identical or orthogonal. Then, errors from different degeneracy classes take the code to corrupted spaces that are mutually orthogonal. Also, for these codes, a non-degenerate code is always pure. In terms of the error correction condition (5), we have $C_{E_1,E_2} = 0$ for errors E_1, E_2 in different degeneracy classes and $C_{E_1,E_2} \neq 0$ for errors in the same degeneracy class.

C. Stabilizer codes

Stabilizer codes [16] are a well known family of quantum error-correcting codes that are analogous to classical linear codes. An $[[n, k, d]]$ stabilizer code maps a 2^k-dimensional k-qubit state space into a 2^k-dimensional subspace of an n-qubit state space.

The code is defined as the space stabilized by an Abelian subgroup of the n-qubit Pauli group, $\mathscr{S} \subset \mathscr{P}_n$, with $n-k$ Hermitian generators, $\mathscr{S} = \langle G_1, \ldots, G_{n-k}\rangle$. For such a space to exist, it is necessary that $-\mathbb{1} \notin \mathscr{S}$. The Abelian group $\mathscr{S}$ is called the *stabilizer* of $\mathcal{Q}$. Explicitly,

$$\mathcal{Q} \equiv \{|\psi\rangle : S|\psi\rangle = |\psi\rangle, \, \forall\, S \in \mathscr{S}\}\,. \tag{6}$$

The code $\mathcal{Q}$ is stabilized by $\mathscr{S}$ iff it is stabilized by all $n-k$ generators G_i. In other words, it is an intersection of subspaces stabilized by G_i,

$$\mathcal{Q} = \bigcap_{i=1}^{n-k} \mathcal{P}(G_i). \tag{7}$$

The *normalizer* of $\mathscr{S}$ in $\mathscr{P}_n$, denoted as $\mathscr{N}$, is the group of all Pauli operators U which fix $\mathscr{S}$ under conjugation ($U^{\dagger} S U = S$ for all $S \in \mathscr{S}$). The term *normalizer* reflects the fact that these operators commute with $\mathscr{S}$ [16]. It is possible to construct $2k$ logical operators $\overline{X}_j$, $\overline{Z}_j$, $j = 1, \ldots, k$, with the usual commutation relations, that together with the generators of $\mathscr{S}$ generate the normalizer (modulo an overall phase factor) [16, 33]. The $(n-k)$ generators G_i of the stabilizer, along with the k operators $\overline{Z}_j$, generate a subgroup $\mathscr{S} \equiv \langle G_1, \ldots, G_{n-k}, \overline{Z}_1, \ldots \overline{Z}_k\rangle$ of $\mathscr{P}_n$ which becomes a maximal Abelian subgroup when including the generator $i\mathbb{1}$. The group $\mathscr{S}$ stabilizes a unique state

$$|s\rangle \equiv |\overline{0 \ldots 0}\rangle. \tag{8}$$

The operators $\overline{X}_j$ acting on $|s\rangle$ generate the basis of the code,

$$|\overline{c_1 \ldots c_k}\rangle \equiv \overline{X}_1^{c_1} \ldots \overline{X}_k^{c_k} |s\rangle. \tag{9}$$

Generally, each detectable Pauli error $E_j \in \mathscr{P}_n$ that acts non-trivially on the code anti-commutes with at least one generator G_i, and errors from different degeneracy classes anti-commute with different subsets of $\mathscr{S}$. We can thus identify a degeneracy class by the set of generators G_i which anti-commute with it. The corrupted code

space $E_j(\mathcal{Q}) = \{E_j|\psi\rangle : |\psi\rangle \in \mathcal{Q}\}$ is anti-stabilized by those generators G_i that anti-commute with E_j. Indeed,

$$G_i(E_j|\psi\rangle) = (-E_jG_i)|\psi\rangle = -E_j|\psi\rangle,$$

which means that the measurement G_i of gives -1. By measuring all G_i, we get the *syndrome* that consists of $n-k$ numbers 1 or -1. There are in total 2^{n-k} possible syndromes identifying different error degeneracy classes, including the trivial error $\mathbb{1}$ which corresponds to the all-one syndrome vector.

Any two corrupted spaces $E_i(\mathcal{Q})$ and $E_j(\mathcal{Q})$ are mutually orthogonal or identical. The whole 2^n-dimensional n-qubit state space $\mathcal{H}_2^{\otimes n}$ is thus divided into $L \equiv 2^{n-k}$ orthogonal 2^k-dimensional subspaces $\mathcal{Q}_j \equiv E_j(\mathcal{Q})$,

$$\mathcal{H}_2^{\otimes n} = \bigoplus_{j=0}^{L-1} \mathcal{Q}_j, \quad \mathcal{Q}_i \perp \mathcal{Q}_j \text{ for } i \neq j. \tag{10}$$

The representatives of different error classes can be chosen to commute with each other and with the logical operations $\overline{X}_i$. These representatives form an Abelian group [34] $\mathscr{T} \equiv \langle g_1, \dots g_{n-k}\rangle$ whose generators g_i can be chosen to anti-commute with only one of the generators of the stabilizer each, $g_iG_j = (-1)^{\delta_{ij}}G_jg_i$ (this follows from Proposition 10.4 in Ref. [2]). Altogether, the generators $\{g_1, \dots, g_{n-k}\}$ can be regarded as a set of Pauli operators forming the basis of the cosets of the normalizer $\mathscr{N}$ of the code $\mathcal{Q}$ in $\mathscr{P}_n$.

Example 1. The $[[5, 1, 3]]$ stabilizer code is defined by the generators

$$\begin{aligned} G_1 =& XZZXI, \quad G_2 = IXZZX, \\ G_3 =& XIXZZ, \quad G_4 = ZXIXZ. \end{aligned} \tag{11}$$

For this code, the logical operators can be taken as

$$\overline{X} = ZZZZZ, \quad \overline{Z} = XXXXX. \tag{12}$$

A basis of the code space is (up to normalization)

$$|\bar{0}\rangle = \prod_{i=1}^{4}(\mathbb{1} + G_i)|00000\rangle, \quad |\bar{1}\rangle = \overline{X}|\bar{0}\rangle.$$

By construction, both basis states are stabilized by the generators G_i. The corresponding stabilizer group is $\mathscr{S} = \langle G_1, \dots, G_4\rangle$. The group of equivalence classes of correctable errors is generated by the representatives (note the mixed notation, e.g., $Z_1Z_3 \equiv ZIZII$)

$$g_1 = Z_1Z_3,\ g_2 = ZZZZI,\ g_3 = ZZIZZ,\ g_4 = Z_2Z_5. \tag{13}$$

The g_j are chosen to commute with the logical operators and also to satisfy $G_ig_j = (-1)^{\delta_{ij}}g_jG_i$. Note that the operators of weight one forming the correctable error set do not by themselves form a group. The generators g_i can be used to map correctable errors to the corresponding group elements with the same syndrome. This gives, e.g., $Z_2 \to g_2g_4 = Z_2\overline{X}$, $X_2 \to g_1 = Z_1Z_3$, $Y_2 \to g_1g_2g_4 = Z_1Z_2Z_3\overline{X} = Z_4Z_5$. □

D. Union Stabilizer codes

The decomposition (10) can be viewed as a constructive definition of the Abelian group $\mathscr{T} \equiv \langle g_1, \dots g_{n-k}\rangle$ of all *translations* of the original stabilizer code $\mathcal{Q}$ in $\mathscr{P}_n$. (In the following, this stabilizer code is denoted $\mathcal{Q}_0$.) Any two non-equivalent translations $t_i, t_j \in \mathscr{T}$ belong to different cosets of the normalizer $\mathscr{N}_0$ of the code $\mathcal{Q}_0$ in $\mathscr{P}_n$ and, therefore, yield mutually orthogonal shifts

$$t_i(\mathcal{Q}_0) \perp t_j(\mathcal{Q}_0), \quad i \neq j. \tag{14}$$

A union stabilizer code [23, 24, 28] (USt) is a direct sum

$$\mathcal{Q} \equiv \bigoplus_{i=1}^{K} t_i(\mathcal{Q}_0) \tag{15}$$

of K shifts of the code $\mathcal{Q}_0$ by non-equivalent translations $t_i \in \mathscr{T}$, $i = 1, \dots K$.

The basis of the code defined by (15) is the union of the sets of basis vectors of all $t_i(\mathcal{Q}_0)$. As a result, the dimension of $\mathcal{Q}$ is $K\,2^k$. This code is then denoted $((n, K2^k, d))$, where d is the distance of the new code. Generally, this distance does not exceed the distance of the original code with respect to the same error set, $d(\mathcal{Q}) \leq d(\mathcal{Q}_0)$. However, if the code $\mathcal{Q}$ is degenerate and the original code $\mathcal{Q}_0$ is one-dimensional, this need not be true.

E. Graph states

The unique state $|s\rangle$ defined by Eq. (8) itself forms a single-state stabilizer code $[[n, 0, d']]$. Its stabilizer $\mathscr{S} \equiv \langle G_1, \dots, G_{n-k}, \overline{Z}_1, \dots \overline{Z}_k\rangle$ has exactly n mutually commuting generators. Note that for stabilizer states we follow the convention that the distance d' is given by the minimum weight of the non-trivial elements of the stabilizer $\mathscr{S}$. Generally, such a state stabilized by an Abelian subgroup in $\mathscr{P}_n$ of order 2^n (which does not include $-\mathbb{1}$) is called a *stabilizer state* [16].

A *graph state* [22] is a stabilizer state of a group whose n mutually commuting generators S_i are defined by a (simple) n-vertex graph $\mathcal{G}$ with an adjacency matrix $R \in \{0, 1\}^{n\times n}$. Specifically, the generators are

$$S_i \equiv X_iZ_1^{R_{i1}}Z_2^{R_{i2}} \dots Z_n^{R_{in}} \equiv X_iZ^{\mathbf{r}_i}, \tag{16}$$

where $\mathbf{r}_i$, $i = 1, \dots n$ denotes the ith row of the adjacency matrix R. A graph state can be efficiently generated [20, 35] by first initializing every qubit in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ (e.g., applying the Hadamard gate on $|0\rangle$), and then using a controlled-Z gate $C_{i,j}^Z$ on every pair (i, j) of qubits corresponding to an edge of the graph $\mathcal{G} \equiv (V, E)$,

$$|s\rangle = \prod_{(i,j)\in E} C_{i,j}^Z H^{\otimes n}|0\rangle_n \equiv U_{\mathcal{G}}|0\rangle_n, \tag{17}$$

where $|0\rangle_n$ is a state with all n qubits in state $|0\rangle$.

Any stabilizer state is locally Clifford-equivalent (LC-equivalent) to a graph state [36–38]. That is, any stabilizer state can be transformed into a graph state by individual discrete rotations of the qubits. This defines the canonical form of a stabilizer state.

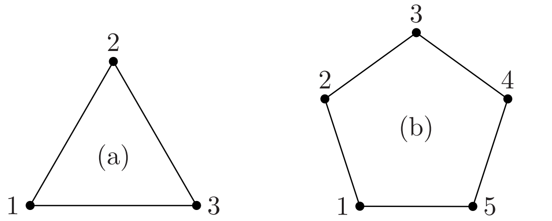

FIG. 2: Ring graphs with (a) 3 vertices and (b) 5 vertices.

Example 2. Consider the ring graph for $n = 3$ [Fig. 2(a)] which defines the stabilizer generators $S_1 = XZZ$, $S_2 = ZXZ$, $S_3 = ZZX$. The corresponding stabilizer state $|s\rangle$ is an equal superposition of all 2^3 states [result of the Hadamard gates in Eq. (17)], taken with positive or negative signs depending on the number of pairs of ones at positions connected by the edges of the graph [result of the gates $C^Z_{i,j} = (-1)^{ij}$]. In the following expressions we omitted normalization for clarity:

$$\begin{aligned} |s\rangle &= |000\rangle + |001\rangle + |010\rangle - |011\rangle \\ &\quad + |100\rangle - |101\rangle - |110\rangle - |111\rangle \qquad (18) \\ &= S_2|s\rangle = |010\rangle - |011\rangle + |000\rangle + |001\rangle \\ &\quad - |110\rangle - |111\rangle + |100\rangle - |101\rangle. \qquad (19) \end{aligned}$$

F. CWS codes and their standard form

Codeword stabilized (CWS) codes [19, 20] represent a general class of non-additive quantum codes that also includes stabilizer codes. They can be viewed as USt codes originating from a stabilizer state. Any CWS code is locally Clifford-equivalent to a CWS code which originates from a graph state [20].

Specifically, a CWS code $((n, K, d))$ in *standard form* [20] is defined by a graph $\mathcal{G}$ with n vertices and a classical code $\mathcal{C}$ containing K binary codewords $\mathbf{c}_i$. The originating stabilizer state is the graph state $|s\rangle$ defined by $\mathcal{G}$, and the *codeword operators* [translations in Eq. (15)] have the form $W_i \equiv Z^{\mathbf{c}_i}$, $i = 1, \ldots K$. For a CWS code in standard form we use the notation $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$.

An important observation made in Ref. [20] is that a single qubit error X, Z, or Y acting on a code state

$$|w_i\rangle \equiv W_i|s\rangle \qquad (20)$$

is equivalent (up to a phase) to an error composed only of Z operators. This establishes the following mapping between multi-qubit errors and classical binary errors. First, let $E_i = Z_i^{\{0,1\}} X_i$ be an error acting on the ith qubit of $W_j|s\rangle$. Then we see that

$$E_i W_j|s\rangle = E_i W_j S_i|s\rangle = \pm(E_i S_i) W_j|s\rangle,$$

where the term $E_i S_i = Z_i^{\{0,1\}} Z^{\mathbf{r}_i}$ consists only of operators Z. The general mapping of an error $E = i^m Z^{\mathbf{v}} X^{\mathbf{u}}$ from the error set $\mathcal{E} \subset \mathscr{P}_n$ to a classical error vector in $\{0, 1\}^n$ is defined as

$$\mathrm{Cl}_{\mathcal{G}}(E) \equiv \mathbf{v} \oplus \bigoplus_{l=1}^{n} u_l \mathbf{r}_l, \qquad (21)$$

where u_l is the lth component of the vector $\mathbf{u}$. We will refer to both the binary vector $\mathrm{Cl}_{\mathcal{G}}(E)$ and the operator $Z^{\mathrm{Cl}_{\mathcal{G}}(E)}$ as the *graph image* of the Pauli operator E.

Theorem 3 from Ref. [20] establishes the correspondence between the error-correcting properties of a quantum code $\mathcal{Q}$ and those of the corresponding classical code $\mathcal{C}$. It states that a CWS code (in standard form) given by a graph $\mathcal{G}$ and word operators $\{W_l = Z^{\mathbf{c}}\}_{\mathbf{c}\in\mathcal{C}}$ detects errors in the set $\mathcal{E}$ if and only if the classical code $\mathcal{C}$ detects errors from the set $\mathrm{Cl}_{\mathcal{G}}(\mathcal{E})$, and for each $E \in \mathcal{E}$,

$$\text{either } \ \mathrm{Cl}_{\mathcal{G}}(E) \ \neq \ \mathbf{0}, \qquad (22)$$

$$\text{or, for each } i, \ \ Z^{\mathbf{c}_i} E \ = \ E Z^{\mathbf{c}_i}. \qquad (23)$$

The code $\mathcal{Q}$ is non-degenerate (and also pure, see Appendix A) iff condition (22) is satisfied for all errors in $\mathcal{E}$ [20, 21]. For a degenerate code, condition (23) needs to be ensured for errors that do not satisfy Eq. (22).

The beauty of the CWS construction is that, for a given code in standard form, we no longer need to worry about possible degeneracies. The classical error patterns induced by the function $\mathrm{Cl}_{\mathcal{G}}(\cdot)$ also separate the errors into corresponding degeneracy classes [20, 21].

The CWS framework is general enough to also include all stabilizer codes [20]. Specifically, a stabilizer code $[[n, k, d]]$ with the stabilizer $\mathscr{S}_0 = \langle G_1, \ldots G_{n-k}\rangle$ and logical operators $\overline{Z}_i$, $\overline{X}_i$, corresponds to a CWS code with the stabilizer

$$\mathscr{S} = \langle G_1, \ldots G_{n-k}, \overline{Z}_1, \ldots, \overline{Z}_k\rangle \qquad (24)$$

and the codeword operator set $\mathscr{W} = \langle \overline{X}_1, \ldots, \overline{X}_k\rangle$ forming a group of size $K = 2^k$. Generally, an LC transformation is required to obtain standard form of this code.

Conversely, an additive CWS code $\mathcal{Q} = ((n, K, d))$ where the codeword operators form an Abelian group (in which case $K = 2^k$ with integer k) is a stabilizer code $[[n, k, d]]$ [20]. In Sec. V B we show that the $n - k$ generators G_i of the stabilizer can be obtained from the graph-state generators S_i [Eq. (16)] by a symplectic Gram-Schmidt orthogonalization procedure [33] which has no effect on the codeword operators.

Example 3. Consider a non-degenerate CWS code $((5, 6, 2))$. The corresponding ring graph is illustrated in Fig. 2(b) (Ref. [20]). The $n = 5$ generators of the stabilizer $\mathscr{S}$ are $S_1 = XZIIZ$ and its four cyclic permutations. The corresponding stabilizer state has a structure similar to Eq. (18), but with more terms. Word operators have the form $Z^{\mathbf{c}_i}$, with the classical codewords

$$\begin{aligned} &\mathbf{c}_0 = 00000, \quad \mathbf{c}_1 = 01101, \quad \mathbf{c}_2 = 10110, \\ &\mathbf{c}_3 = 01011, \quad \mathbf{c}_4 = 10101, \quad \mathbf{c}_5 = 11010. \end{aligned} \qquad (25)$$

□

Example 4. To express the $[[5, 1, 3]]$ stabilizer code [see Example 1] as a $((5, 2, 3))$ CWS code in standard form, we explicitly construct alternative generators S_i of the stabilizer $\mathscr{S} = \langle G_1, G_2, G_3, G_4, \overline{Z}\rangle$ [Eq. (24)] to contain only one X operator each. We obtain $S_3 = G_1G_2\overline{Z} = IZXZI$ and its four cyclic permutations. This does not require any qubit rotations due to a slightly unconventional choice of the logical operators in Eq. (12). The corresponding graph is the ring [20], see Fig. 2(b). The codeword operators are $\mathscr{W} = \{I, \overline{X}\}$, which by Eq. (12) correspond to classical binary codewords $\{00000, 11111\}$. Note that the error map induced by the graph is different from the mapping to group elements in Example 1; in particular, $Z^{\mathrm{Cl}_\mathcal{G}(Z_2)} = Z_2$, $Z^{\mathrm{Cl}_\mathcal{G}(X_2)} = Z_1Z_3$, $Z^{\mathrm{Cl}_\mathcal{G}(Y_2)} = Z_1Z_2Z_3$. □

III. GENERIC RECOVERY FOR CWS CODES

In this section we construct a generic recovery algorithm which can be adapted to any non-additive code. To our best knowledge, such an algorithm has not been explicitly constructed before.

The basic idea is to construct a non-Pauli measurement operator $M_\mathcal{Q} = 2P_\mathcal{Q} - \mathbb{1}$, where $P_\mathcal{Q} \equiv \sum_{i=1}^K |w_i\rangle\langle w_i|$ is the projector onto the code $\mathcal{Q}$ spanned by the orthonormal basis $\{|w_i\rangle\}_{i=1}^K$. The measurement operator is further decomposed using the identity

$$-M_\mathcal{Q} = \mathbb{1} - 2\sum_i^K |w_i\rangle\langle w_i| = \prod_i^K (\mathbb{1} - 2|w_i\rangle\langle w_i|). \quad (26)$$

We use the graph state encoding unitary $U_\mathcal{G}$ from Eq. (17) and the following decomposition [20] of the standard-form basis of the CWS code in terms of the graph state $|s\rangle$ and the classical states $|c_i\rangle$:

$$|w_i\rangle = Z^{\mathbf{c}_i}|s\rangle = Z^{\mathbf{c}_i}U_\mathcal{G}|0\rangle_n = U_\mathcal{G}X^{\mathbf{c}_i}|0\rangle_n = U_\mathcal{G}|\mathbf{c}_i\rangle.$$

The measurement operator $M_\mathcal{Q}$ is rewritten as a product

$$M_\mathcal{Q} = -U_\mathcal{G}M_\mathcal{C}U_\mathcal{G}^\dagger, \quad M_\mathcal{C} \equiv \prod_{i=1}^K M_{\mathbf{c}_i} \quad (27)$$

where the measurement operator

$$M_{\mathbf{c}_i} \equiv \mathbb{1} - 2|\mathbf{c}_i\rangle\langle\mathbf{c}_i| = X^{\overline{\mathbf{c}}_i}\big(\mathbb{1} - 2|1\rangle_n\langle 1|_n\big)X^{\overline{\mathbf{c}}_i}, \quad (28)$$

stabilizes the orthogonal complement of the classical state $|\mathbf{c}_i\rangle$. The components of the binary vector $\overline{\mathbf{c}}_i$ are the respective complements of $\mathbf{c}_i$, $\overline{c}_{i,j} = 1 \oplus c_{i,j}$. The operator in the parentheses in Eq. (28) is the $(n-1)$-controlled Z gate (the Z-operator is applied to the n-th qubit only if all the remaining qubits are in state $|1\rangle$). It can also be represented as n-qubit controlled phase gate $C_n^\theta \equiv \exp(i\theta P_{|1\rangle_n})$ with $\theta = \pi$, where the operator $P_{|1\rangle_n}$ projects onto the state $|1\rangle_n$. This can be further decomposed as a product of two Hadamard gates and an $(n-1)$-controlled CNOT gate [Fig. 3] which for $n \geq 6$ can be implemented in terms of one ancilla and $8(n-4)$ three-qubit Toffoli gates [39] and therefore has linear complexity in n. With no ancillas, the complexity of the $(n-1)$-controlled CNOT gate is $\mathcal{O}(n^2)$ [39].

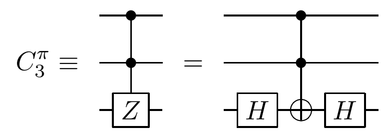

FIG. 3: Decomposition of n-qubit controlled-Z gate C_n^Z in terms of $(n-1)$-controlled CNOT gate (n-qubit Toffoli gate) for $n = 3$.

The corresponding ancilla-based measurement for $M_{\mathbf{c}_i}$ can be constructed with the help of two Hadamard gates [Fig. 1] by adding an extra control to each C_n^π gate. Indeed, this correlates the state $|1\rangle$ of the ancilla with $M_{\mathbf{c}_i}$ acting on the n qubits, and the state $|0\rangle$ of the ancilla with $\mathbb{1}$. When constructing the measurement for the product of the operators $M_{\mathbf{c}_i}$, it is sufficient to use only one ancilla, since for each basis state $|w_i\rangle$ only one of these operators acts non-trivially. The classical part of the overall measurement circuit without the graph state encoding $U_\mathcal{G}$ is shown in Fig. 4.

The complexity of measuring $M_\mathcal{Q}$ [Eq. (27)] then becomes K times the complexity of $(n+1)$-qubit Toffoli gate for measuring each $M_{\mathbf{c}_i}$, plus the complexity of the encoding circuit $U_\mathcal{G}$ and its inverse $U_\mathcal{G}^\dagger$, which is at most n^2. Overall, for large n, the measurement complexity is no larger than $n^2 + K\mathcal{O}(n)$, or $(1+K)\mathcal{O}(n^2)$ for a circuit without additional ancillas.

We would like to emphasize that so far we have only constructed the measurement for *error detection*. Actual *error correction* for a non-additive code in this scheme involves constructing measurements $M_E \equiv EM_\mathcal{Q}E^\dagger$ for all corrupted subspaces corresponding to different degeneracy classes given by different $\mathrm{Cl}_\mathcal{G}(E)$. This relies on the orthogonality of the corrupted subspaces, see Appendix A. For a general t-error correcting code, the number of these measurements can reach the same exponential order $B(n, t)$ as the number of correctable errors in Eq. (1). For non-degenerate codes, we cannot do better using this method.

Note that the measurement circuit derived in this section first decodes the quantum information, then performs the measurement for the classical code, and finally re-encodes the quantum state.

IV. MEASUREMENTS FOR USt CODES

In this section we construct a quantum circuit for the measurement operator $M_\mathcal{Q}$ of a USt code $((n, K2^k, d))$. To this end, we define the logical combinations of non-Pauli measurements in agreement with analogous combinations defined in Ref. [30] for the projection operators,

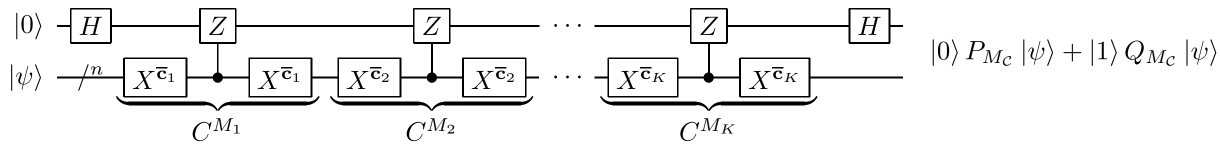

FIG. 4: Generic measurement of the classical part $M_\mathcal{C}$ of the CWS code stabilizer $M_\mathcal{Q}$ [Eq. (27)] uses K $(n+1)$-qubit controlled-Z gates. Here $X^{\bar{\mathbf{c}}_i}$ indicates that a single-qubit gate $X^{\bar{\mathbf{c}}_{i,j}}$ is applied at the j th qubit, $j = 1, \ldots n$, and C^{M_i} is the controlled-$M_{\mathbf{c}_i}$ gate [see Eq. (28)]. This can be further simplified by combining the neighboring $X^{\bar{c}}_{i,j}$-gates and replacing the controlled-Z gates by $(n+1)$-qubit Toffoli gates as in Fig. 3.

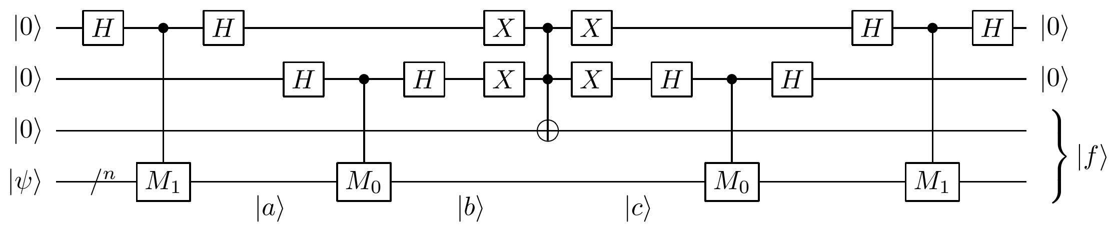

FIG. 5: Measurement $M_1 \wedge M_0$ by performing logical AND operation on two ancillas. We use the notations $P_i \equiv P_{M_i}$, $Q_i \equiv \mathbb{1} - P_i$, $i = 0, 1$ and assume $P_1 P_0 = P_0 P_1$. The circuit returns $|f\rangle = |1\rangle P_1 P_0 |\psi\rangle + |0\rangle (\mathbb{1} - P_1 P_0) |\psi\rangle$. Intermediate results are: $|a\rangle = |000\rangle P_1 |\psi\rangle + |100\rangle Q_1 |\psi\rangle$, $|b\rangle = |000\rangle P_1 P_0 |\psi\rangle + |010\rangle P_1 Q_0 |\psi\rangle + |100\rangle Q_1 P_0 |\psi\rangle + |110\rangle Q_1 Q_0 |\psi\rangle$, $|c\rangle = |001\rangle P_1 P_0 |\psi\rangle + |010\rangle P_1 Q_0 |\psi\rangle + |100\rangle Q_1 P_0 |\psi\rangle + |110\rangle Q_1 Q_0 |\psi\rangle$; the last two groups of gates return the first two ancillas to the state $|00\rangle$.

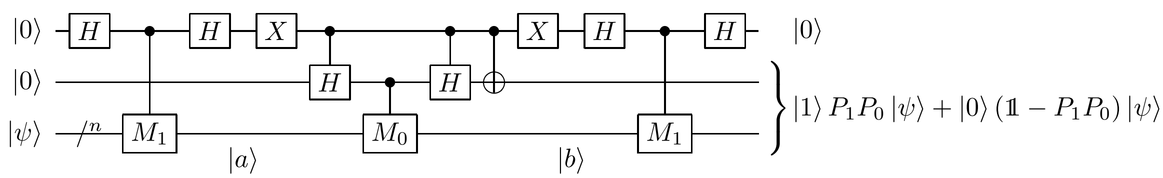

FIG. 6: Simplified measurement for $M_1 \wedge M_0$; notations as in Fig. 5. Intermediate results (cf. Fig. 1) are: $|a\rangle = |00\rangle P_1 |\psi\rangle + |10\rangle Q_1 |\psi\rangle$, $|b\rangle = |0\rangle (|1\rangle P_0 + |0\rangle Q_0) P_1 |\psi\rangle + |10\rangle Q_1 |\psi\rangle$. The effect of the last block is to select the component with the first ancilla in the state $|0\rangle$; this requires that the projectors P_1 and P_0 commute. The result is equivalent to $|1\rangle P_{M_1 \wedge M_0} |\psi\rangle + |0\rangle (\mathbb{1} - P_{M_1 \wedge M_0}) |\psi\rangle$.

and construct the circuits for logical combinations AND [Figs. 5, 6] and XOR [Figs. 7, 8]. We use these circuits to construct the measurement for $M_\mathcal{Q}$ with complexity not exceeding $2K(n+1)(n-k)$.

A. Algebra of measurements

Logical AND: Given two commuting measurement operators M_1 and M_0, let $M_1 \wedge M_0$ denote the measurement operator that stabilizes all states in the subspace

$$\mathcal{P}(M_1 \wedge M_0) \equiv \mathcal{P}(M_1) \cap \mathcal{P}(M_0). \tag{29}$$

The output of the measurement $M_1 \wedge M_0$ is identical to the logical AND operation performed on the output of measurements M_1 and M_0. This measurement can be implemented by the circuit in Fig. 5. Here the first two ancillas are entangled with the two measurement outcomes; the third ancilla is flipped only if both ancillas are in the $|0\rangle$ state, which gives the combination $|1\rangle P_{M_1 \wedge M_0} |\psi\rangle$.

The projector onto the positive eigenspace of $M_1 \wedge M_0$ satisfies the identity

$$P_{M_1 \wedge M_0} = P_{M_1} P_{M_0}. \tag{30}$$

This identity can be used to obtain a simplified circuit which only uses two ancillas, see Fig. 6, with the price of two additional controlled-Hadamard gates [Fig. 9].

The circuits in Figs. 5 and 6 can be generalized to perform the measurement corresponding to the logical AND of $\ell > 1$ commuting measurement operators with the help of associativity, e.g., $M_2 \wedge M_1 \wedge M_0 = M_2 \wedge (M_1 \wedge M_0)$. The generalization of the simplified circuit in Fig. 6 requires only two ancillas for any $\ell > 1$. The corresponding complexity is $2\ell - 1$ times the complexity of a controlled-M gate, plus $2\ell - 1$ times the complexity of a controlled single-qubit gate. When all M_i are n-qubit Pauli operators, the overall complexity with two ancillas is $(2\ell - 1)(n + 1)$.

Logical XOR: In analogy to the logical "exclusive OR", we define the symmetric difference $A \triangle B \triangle C \ldots$ of vector spaces A, B, C, $\ldots$ as the vector space formed by the basis vectors that belong to an odd number of the original

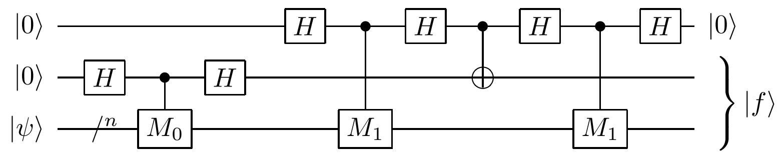

FIG. 7: Measurement $M_1 \boxplus M_0$ by performing logical XOR gate on the two ancillas and subsequent recovery of the first ancilla. Notations as in Fig. 5. The final result is $|f\rangle = |1\rangle\,(P_0Q_1 + P_1Q_0)\,|\psi\rangle + |0\rangle\,(P_0P_1 + Q_0Q_1)\,|\psi\rangle$.

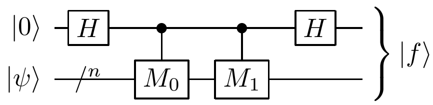

FIG. 8: Simplified measurement for $M_1 \boxplus M_0$. Notations as in Fig. 5. The result (cf. Fig. 1) $|f\rangle = |1\rangle\,(Q_1P_0 + P_1Q_0)\,|\psi\rangle + |0\rangle\,(P_1P_0 + Q_1Q_0)\,|\psi\rangle$ is equivalent to $|1\rangle\, P_{M_1 \boxplus M_0}\,|\psi\rangle + |0\rangle\,(\mathbb{1} - P_{M_1 \boxplus M_0})\,|\psi\rangle$.

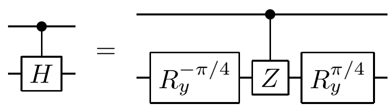

FIG. 9: Implementation of the controlled-H gate based on the identity $H = \exp(-iY\pi/8)Z\exp(iY\pi/8)$.

vector spaces. For two vector spaces

$$A \bigtriangleup B \equiv (A \cap B^\perp) \oplus (B \cap A^\perp). \tag{31}$$

This operation is obviously associative, $A \bigtriangleup (B \bigtriangleup C) = (A \bigtriangleup B) \bigtriangleup C$. For two commuting measurement operators M_0, M_1, let $M_1 \boxplus M_0$ be the measurement operator that stabilizes the subspace $\mathcal{P}(M_1) \bigtriangleup \mathcal{P}(M_0)$. Explicitly,

$$\begin{aligned} \mathcal{P}(M_1 \boxplus M_0) &\equiv \mathcal{P}(M_1) \bigtriangleup \mathcal{P}(M_0) \\ &= \left[\mathcal{P}(M_1) \cap \mathcal{P}^\perp(M_0)\right] \oplus \left[\mathcal{P}(M_0) \cap \mathcal{P}^\perp(M_1)\right]. \end{aligned} \tag{32}$$

The output of measuring $M_1 \boxplus M_0$ is identical to the logical XOR operation performed on the outputs of measurements M_1 and M_0. The corresponding measurement can be implemented by combining the two ancillas with a CNOT gate [Fig. 7].

To simplify this measurement, we show that $M_1 \boxplus M_0 = -M_1M_0$. Indeed, Eq. (32) implies that for the projection operators $P_i \equiv P_{M_i}$, $i = 0, 1$,

$$P_{M_1 \boxplus M_0} = P_1(\mathbb{1} - P_0) + P_0(\mathbb{1} - P_1).$$

The corresponding measurement operator is factorized with the help of the projector identities (3),

$$\begin{aligned} M_1 \boxplus M_0 &= 2[P_1(\mathbb{1} - P_0) + P_0(\mathbb{1} - P_1)] - \mathbb{1} \\ &= -(2P_1 - \mathbb{1})(2P_0 - \mathbb{1}) = -M_1M_0. \end{aligned}$$

This implies that $\mathcal{P}(M_1 \boxplus M_0) = \mathcal{P}^\perp(M_1M_0)$. In other words, the measurement of $M_1 \boxplus M_0$ can be implemented simply as an (inverted) concatenation of two measurements, see Fig. 8. The same circuit can also be obtained from that in Fig. 7 by a sequence of circuit simplifications (not shown).

The circuit in Fig. 8 is immediately generalized to a combination of more than two measurements, $M_1 \boxplus \ldots \boxplus M_\ell = (-1)^{l-1} M_1 \ldots M_\ell$. The corresponding complexity for computing the XOR of ℓ measurements is simply the sum of the individual complexities, implying that this concatenation has no overhead.

B. Error detection for union stabilizer codes

A USt code $\mathcal{Q} = ((n, K\,2^k, d))$ is a direct sum (15) of K mutually orthogonal subspaces obtained by translating the originating stabilizer code $\mathcal{Q}_0 = [[n, k, d_0]]$. For mutually orthogonal subspaces $A \perp B$, we have $A \subset B^\perp$ and $B \subset A^\perp$, and the direct sum is the same as the symmetric difference (31), $A \oplus B = A \bigtriangleup B$.

In turn, the stabilizer code $\mathcal{Q}_0$ is an intersection of the subspaces stabilized by the generators G_i of the stabilizer, see Eq. (7). The translated subspaces $t_j(\mathcal{Q}_0)$ are stabilized by the Pauli operators $M_{i,j} = t_j G_i t_j^\dagger$. We can therefore decompose the USt code $\mathcal{Q}$ as

$$\mathcal{Q} = \bigoplus_{j=1}^{K} \left[\bigcap_{i=1}^{n-k} \mathcal{P}(M_{i,j})\right] = \mathop{\bigtriangleup}_{j=1}^{K} \left[\bigcap_{i=1}^{n-k} \mathcal{P}(M_{i,j})\right]. \tag{33}$$

This gives the decomposition of the measurement operator $M_\mathcal{Q}$ whose positive eigenspace is the code $\mathcal{Q}$ as

$$M_\mathcal{Q} = \mathop{\boxplus}_{j=1}^{K} \left[\bigwedge_{i=1}^{n-k} M_{i,j}\right]. \tag{34}$$

Recall that the complexity of each of the K logical AND operations is $[2(n-k)-1](n+1)$. No additional overhead is required to form the logical XOR of the results. Thus, we obtain the following

Theorem 1 *Error detection for a USt code of length n and dimension $K\,2^k$, formed by a translation set of size $K > 0$, has complexity at most $2K(n-k)(n+1)$.*

Note that in the special case of a CWS code ($k = 0$), the prefactor of K is quadratic in n whereas the corresponding prefactor obtained in Sec. III is *linear* in n. The reason is that in Eq. (27) the graph encoding circuit $Q_\mathcal{G}$ with complexity $\mathcal{O}(n^2)$ is used only twice, and the projections onto the classical states have linear complexity. In Eq. (34) we are using K projections onto basis states

of the quantum code. The advantage of the more complex measurement constructed in this section is that it does not involve having unprotected decoded qubits for the entire duration of the measurement.

V. STRUCTURED MEASUREMENT FOR CWS CODES

A. Grouping correctable errors

Recall from Section II C that for stabilizer codes the representatives of the error degeneracy classes form an Abelian group whose generators are in one-to-one correspondence with the generators of the stabilizer. Measuring the $n-k$ generators of the stabilizer of a stabilizer code $[[n,k,d]]$ uniquely identifies the degeneracy class of the error.

In this section we establish a similar structure for CWS codes. First, for any subset $\mathcal{D}\subset\mathcal{E}$ of correctable errors of a quantum code $\mathcal{Q}$, we define the set $\mathcal{E}_{\overline{\mathcal{D}}}$ of *unrelated* errors which do not fall in the same degeneracy class with any error from $\mathcal{D}$. The formal definition

$$\mathcal{E}_{\overline{\mathcal{D}}}\equiv\{E: E\in\mathcal{E}\wedge C_{E,E_1}=0, \forall E_1\in\mathcal{D}\} \qquad (35)$$

is based on the general error correction condition (5) and the orthogonality of corrupted spaces, see Appendix A. When errors in $\mathcal{E}$ are non-degenerate, the definition (35) is equivalent to the set difference, $\mathcal{E}\setminus\mathcal{D}$. In the general case, since we do not distinguish between mutually degenerate errors, $\mathcal{E}_{\overline{\mathcal{D}}}$ can be thought of as the difference between the sets of degeneracy classes in $\mathcal{E}$ and in $\mathcal{D}$.

Definition (35) implies that the subspaces

$$\mathcal{D}(\mathcal{Q})\equiv\bigoplus_{E\in\mathcal{D}}E(\mathcal{Q})$$

and $\mathcal{E}_{\overline{\mathcal{D}}}(\mathcal{Q})$, defined analogously, are mutually orthogonal. Moreover, if the elements of the set $\mathcal{D}$ form a group $\mathscr{D}\equiv\mathcal{D}$, the subspace $\mathscr{D}(\mathcal{Q})$ is also orthogonal to $\mathcal{E}_{\overline{\mathscr{D}}}[\mathscr{D}(\mathcal{Q})]$ [see Eq. (40) below]. In other words, $\mathcal{Q}_{\mathscr{D}}\equiv\mathscr{D}(\mathcal{Q})$ can be viewed as a quantum code which detects errors from $\mathscr{E}_{\overline{\mathscr{D}}}$.

This observation, together with the error-detection measurement for USt codes constructed in the previous section, forms the basis of our error grouping technique. We prove the following

Theorem 2 *For a CWS code $\mathcal{Q}=(\mathcal{G},\mathcal{C})$ in standard form and a group $\mathscr{D}$ formed by graph images of some correctable errors in $\mathcal{E}$, the code $\mathcal{Q}_{\mathscr{D}}\equiv\mathscr{D}(\mathcal{Q})$ is a USt code which detects all errors in $\mathcal{E}_{\overline{\mathscr{D}}}$.*

Proof. First, we show that the subspace $\mathscr{D}(\mathcal{Q})$ is a USt code. The corresponding set of basis vectors is

$$\mathscr{D}\left(\{|w_1\rangle,\ldots,|w_K\rangle\}\right)\equiv\bigcup_{e_\alpha\in\mathscr{D}}\bigcup_{i=1}^{K}\{e_\alpha|w_i\rangle\}. \qquad (36)$$

These vectors are mutually orthogonal,

$$\langle w_i|\,e_\beta^\dagger e_\alpha\,|w_j\rangle=0,\ \forall i,j\le K,\ e_\alpha,e_\beta\in\mathscr{D},\ e_\alpha\ne e_\beta, \qquad (37)$$

since every element $e_\alpha=Z^{\mathrm{Cl}_{\mathcal{G}}(E_\alpha)}$ of the group $\mathscr{D}$ is a representative of a separate error degeneracy class. Further, the group $\mathscr{D}$ is Abelian, and its elements commute with the codeword generators $W_i=Z_i^{\mathbf{c}}$, $c_i\in\mathcal{C}$. Therefore, using Eq. (20), we can rearrange the set (36) as

$$\mathscr{D}\left(\{|w_1\rangle,\ldots,|w_K\rangle\}\right)=\bigcup_{i=1}^{K}W_i\left(\{e_\alpha|s\rangle\}_{e_\alpha\in\mathscr{D}}\right). \qquad (38)$$

The set in the parentheses on the right hand side is a basis of the additive CWS code $\mathcal{Q}_{0\mathscr{D}}$ formed by the group $\mathscr{D}$ acting on the graph state $|s\rangle$. Then, we can write the subspace $\mathscr{D}(\mathcal{Q})$ explicitly as a USt code [cf. Eq. (15)]

$$\mathscr{D}(\mathcal{Q})=\bigoplus_{i=1}^{K}W_i(\mathcal{Q}_{0\mathscr{D}}), \qquad (39)$$

where the translations are given by the set of codeword operators $\mathcal{W}\equiv\{W_i\}_{i=1}^K$ of the original code $\mathcal{Q}$. Orthogonality condition (14) is ensured by Eq. (37).

Second, we check the error-detection condition (4) for the code (39). Explicitly, for an error $E\in\mathcal{E}_{\overline{\mathscr{D}}}$, and for the orthogonal basis states $e_\alpha W_i|s\rangle$,

$$\langle s|\,W_i{}^\dagger e_\alpha^\dagger E e_\beta W_j\,|s\rangle=\pm\langle w_i|\,E e_\alpha^\dagger e_\beta\,|w_j\rangle=0 \qquad (40)$$

for all α,β,i,j, according to Eqs. (5), (35) and the group property of $\mathscr{D}$. □

Now, to correct errors in groups, we just have to find a suitable decomposition of the graph images of the original error set into a collection of groups, $\mathrm{Cl}_{\mathcal{G}}(\mathcal{E})=\bigcup_j\mathscr{D}_j$, and perform individual error-detection measurements for the auxiliary codes $\mathcal{Q}_{\mathscr{D}_j}$ until the group containing the error is identified.

To find an error within a group $\mathscr{D}\equiv\langle g_1,\ldots,g_m\rangle$ with m generators, we can try all m subgroups of $\mathscr{D}$ with one generator missing. More specifically, for a generator g_l we consider the subgroup $\mathscr{D}^{(l)}=\langle g_1,\ldots,g_{l-1},g_{l+1},\ldots,g_m\rangle$ and perform error detection for the code $\mathcal{Q}_{\mathscr{D}}^{(l)}\equiv\mathscr{D}^{(l)}(\mathcal{Q})$. After completing m measurements, we obtain a representative of the actual error class. This is the product of all generators g_l for which the corresponding code $\mathcal{Q}_{\mathscr{D}}^{(l)}$ detected an error.

B. Complexity of a combined measurement

To actually carry out the discussed program, we need to construct the $n-m$ generators G_i of the stabilizer of the code $\mathcal{Q}_{0\mathscr{D}}$. The generators have to commute with the m generators g_α in the group $\mathscr{D}$.

This can be done with the Gram-Schmidt (GS) orthogonalization [33] of the graph-state generators S_i [Eq. (16)]

with respect to the generators g_α. As a result, we obtain the orthogonalized set of independent generators S_i' such that $g_\alpha S_i' = (-1)^{\delta_{i\alpha}} S_i' g_\alpha$. We can take the last $n-m$ of the obtained generators as the generators of the stabilizer, $G_i = S_{i+m}'$, $i = 1,\ldots,n-m$.

The orthogonalization procedure is guaranteed to produce exactly m generators S_α' anti-commuting with the corresponding errors g_α, $\alpha = 1,\ldots,m$. Indeed, the GS orthogonalization procedure can be viewed as a sequence of row operations applied to the original $n \times m$ binary matrix B with the elements $b_{i\alpha}$ which define the original commutation relation,

$$S_i g_\alpha = (-1)^{b_{i\alpha}} g_\alpha S_i. \tag{41}$$

The generator g_α anti-commutes with at least one operator in $\mathscr{S}$ if and only if the α-th column of B is not an all-zero column. Then all m generators are independent (no generator can be expressed as a product of some others) if and only if B has full column rank.

By this explicit construction, the generators G_i of the stabilizer of the auxiliary code $\mathcal{Q}_{\mathscr{D}}$ [Eq. (39)] are Pauli operators in the original graph-state basis. The complexity of each error-detection measurement $M(\mathcal{Q}_{\mathscr{D}})$ is therefore given by Theorem 1.

C. Additive CWS codes

The procedure described above appears to be extremely tedious, much more complicated than the syndrome measurement for a stabilizer code. However, it turns out that for stabilizer codes this is no more difficult than the regular syndrome-based error correction.

Indeed, for a stabilizer code $\mathcal{Q} = [[n,k,d]]$, the degeneracy classes for all correctable errors form a group of all translations of the code, $\mathscr{D} \equiv \mathscr{T} = \langle g_1,\ldots,g_{n-k}\rangle$, with $n-k$ generators. To locate the error, we just have to go over all $n-k$ USt codes $\mathscr{D}^{(l)}(\mathcal{Q})$ generated by the subgroups of $\mathscr{D}$ with the generator g_l missing. Since the originating code $\mathcal{Q}$ is a stabilizer code, the USt codes $\mathscr{D}^{(l)}(\mathcal{Q})$ are actually stabilizer codes, encoding $\tilde{k} = k + (n-k-1) = n-1$ qubits each. Hence there is only one non-trivial error, and up to error degeneracies, $\mathcal{E}_{\mathscr{D}^{(l)}} = \langle g_l\rangle$. The corresponding stabilizers $\mathscr{S}^{(l)}$ have only one generator each. The necessary measurements are just independent measurements of $n-k$ Pauli operators, the same as needed to measure the syndrome. Moreover, if the error representatives g_l, $l = 1,\ldots,n-k$, are chosen to satisfy the orthogonality condition $G_i g_l = (-1)^{\delta_{il}} g_l G_i$ as in Example 1, the operators to be measured are the original generators G_i of the stabilizer, and the corresponding measurement is just the syndrome measurement.

Example 5. Consider the additive code $((5,2,3))$ equivalent to the stabilizer code $[[5,1,3]]$, see Examples 1 and 4. The graph-induced maps of single-qubit errors form a group of translations of the code, $\mathscr{T} = \langle Z_1, Z_2, Z_3, Z_4\rangle$. This group contains all error degeneracy classes, $\mathscr{D} = \mathscr{T}$. With the addition of the logical operator $\overline{X} \equiv ZZZZZ$, these can generate the entire 5-qubit Hilbert space $\mathcal{H}_2^{\otimes 5}$ from the graph state $|s\rangle$; we have $\mathscr{T}(\mathcal{Q}) = \mathcal{H}_2^{\otimes 5}$.

Indeed, if we form a measurement as for a generic CWS code, we first obtain the stabilizer of the auxiliary code $\mathcal{Q}_{0\mathscr{D}}$ [Eq. (39)] which in this case has only one generator, $\mathscr{S} = \langle S_5\rangle$, where $S_5 = ZIIZX$, see Example 4. Translating this code with the set (in this case, group) $\mathscr{W} = \{I, \overline{X}\}$ of codeword operators, we get the auxiliary USt code $\mathscr{W}(\mathcal{Q}_{0\mathscr{D}})$ as the union of the positive eigenspaces of the operators [see Eq. (33)],

$$M_{1,0} = S_5, \quad M_{1,1} = \overline{X} S_5 \overline{X}^\dagger = -S_5,$$

which is the entire Hilbert space, $\mathscr{W}(\mathcal{Q}_{0\mathscr{D}}) = \mathscr{D}(\mathcal{Q}) = \mathcal{H}_2^{\otimes 5}$, as expected.

To locate the error within the group $\mathscr{D}$ with $m = 4$ generators, we form a set of smaller codes $\mathcal{Q}_{\mathscr{D}}^{(l)} \equiv \mathscr{D}^{(l)}(\mathcal{Q})$, $l = 1,\ldots,4$, where the group $\mathscr{D}^{(l)}$ is obtained from $\mathscr{D}$ by removing the l-th generator. The corresponding stabilizers are $\mathscr{S}^{(1)} = \langle S_1, S_5\rangle$, $\mathscr{S}^{(2)} = \langle S_2, S_5\rangle$, etc. The matrices $M_{i,j}^{(l)}$ of conjugated generators have the form, e.g.,

$$M_{i,j}^{(1)} = \begin{pmatrix} S_1, & -S_1 \\ S_5, & -S_5 \end{pmatrix}, \quad M_{i,j}^{(2)} = \begin{pmatrix} S_2, & -S_2 \\ S_5, & -S_5 \end{pmatrix}, \ldots$$

The code $\mathcal{Q}_{\mathscr{D}}^{(l)}$ is formed as the union of the common positive eigenspaces of the operators in the columns of the matrix $M_{i,j}^{(l)}$. Clearly, these codes can be more compactly introduced as positive eigenspaces of the operators $\tilde{G}_l = S_l S_5$, $l = 1,\ldots,4$. Such a simplification only happens when the original code $\mathcal{Q}$ is additive. While the operators $\tilde{G}_i$ are different from the stabilizer generators in Eq. (11), they generate the same stabilizer $\mathscr{S} = \langle \tilde{G}_1,\ldots,\tilde{G}_4\rangle$ of the original code $\mathcal{Q}$. It is also easy to check that the same procedure gives the original generators G_i [Eq. (11)] if we start with the error representatives (13). □

D. Generic CWS codes

Now consider the case of a generic CWS code $\mathcal{Q} = ((n,K,d))$. Without analyzing the graph structure, it is impossible to tell whether there is any set of classical images of correctable errors that forms a large group. However, since we know its minimum distance, we know that the code can correct errors located on $t = \lfloor (d-1)/2 \rfloor$ qubits. All errors located on a given set of qubits form a group. Therefore, by taking an *index set* $A \subset \{1,\ldots,n\}$ of $s \le t$ different qubit positions, we can ensure that the corresponding correctable errors $\{E_j\}_{j=1}^{4^s}$ form a group with $2s$ independent generators. The corresponding graph images $\mathrm{Cl}_{\mathcal{G}}(E_j) \in \mathscr{D}_A$ obey the same multiplication table, but they are not necessarily independent. As a result, the Abelian group $\mathscr{D}_A$ generally

has $m \leq 2s$ generators. Since all group elements correspond to correctable errors, the conditions of Theorem 2 are satisfied.

Overall, to locate an error of weight t or less, we need to iterate over each (but the last one) of the $\binom{n}{t}$ index sets of size t and perform the error-detecting measurements in the corresponding USt codes $\mathcal{E}_A(\mathcal{Q})$ until the index set with the error is found. This requires up to $\binom{n}{t}-1$ measurements to locate the index set, and the error can be identified after additional $m \leq 2t$ measurements. This can be summarized as the following

Theorem 3 *A CWS code of distance d can correct errors of weight up to $t = \lfloor (d-1)/2 \rfloor$ by performing at most*

$$N(n,t) \equiv \binom{n}{t} + 2t - 1 \tag{42}$$

measurements.

For any length $n \geq 3$, this scheme reduces the total number (1) of error patterns by a factor

$$\frac{B(n,t)}{N(n,t)} \geq \begin{cases} \dfrac{3n+1}{n+1}, & \text{if } t = 1, \\ 3^t, & \text{if } t > 1. \end{cases} \tag{43}$$

Example 6. Consider the $((5,6,2))$ code previously discussed in Example 3. While the distance $d = 2$ is too small to correct arbitrary errors, we can correct an error located at a given qubit. Assume that an error may have happened on the second qubit. Then we only need to check the index set $A = \{2\}$. The errors $\{\mathbb{1}, X_2, Y_2, Z_2\}$ located in A form a group with generators $\{X_2, Z_2\}$; the corresponding group of classical error patterns induced by the ring graph in Fig. 2(b) is $\mathscr{D}_A = \langle Z_1 Z_3, Z_2 \rangle$. The three generators G_i of the stabilizer of the originating USt code $\mathcal{Q}_{0\mathscr{D}_A}$ can be chosen as, e.g., $G_1 = S_1 S_3 = XIXZZ$, $G_2 = S_4 = IIZXZ$, $G_3 = S_5 = ZIIZX$. Using the classical codewords (25) for the translation operators $t_j = Z^{\mathbf{c}_j}$, we obtain the conjugated generators $M_{i,j} = t_j G_i t_j^\dagger$

$$M_{i,j} = \begin{pmatrix} G_1, & -G_1, & G_1, & G_1, & G_1, & -G_1 \\ G_2, & G_2, & -G_2, & -G_2, & G_2, & -G_2 \\ G_3, & -G_3, & G_3, & -G_3, & -G_3, & G_3 \end{pmatrix}. \tag{44}$$

According to Eq. (33), the auxiliary code $\mathcal{Q}_{\mathscr{D}_A}$ is a direct sum of the common positive eigenspaces of the operators in the six columns of the matrix (44).

To locate the actual error in this 24-dimensional space, we consider the two subgroups $\mathscr{D}^{(1)} = \langle ZIZII \rangle$ and $\mathscr{D}^{(2)} = \langle IZIII \rangle$ of $\mathscr{D}_A$. The stabilizers of the corresponding auxiliary codes $\mathcal{Q}^{(1)}_{\mathscr{D}_A}$ and $\mathcal{Q}^{(2)}_{\mathscr{D}_A}$ can be obtained by adding $G_4^{(1)} = S_2 = ZXZII$ and $G_4^{(2)} = S_3 = IZXZI$, respectively; this adds one of the rows

$$M^{(1)}_{4,j} = (S_2, -S_2,\ \ S_2, -S_2,\ \ S_2, -S_2), \tag{45}$$

$$M^{(2)}_{4,j} = (S_3, -S_3, -S_3,\ \ S_3, -S_3,\ \ S_3) \tag{46}$$

to the matrix (44). The original code $\mathcal{Q}$ is the intersection of the codes $\mathcal{Q}^{(1)}_{\mathscr{D}_A}$ and $\mathcal{Q}^{(2)}_{\mathscr{D}_A}$; the corrupted space $X_2(\mathcal{Q})$ is located in $\mathcal{Q}^{(1)}_{\mathscr{D}_A}$, but not in $\mathcal{Q}^{(2)}_{\mathscr{D}_A}$, while, e.g., the corrupted space $Y_2(\mathcal{Q})$ is located in $\mathcal{Q}_{\mathscr{D}_A}$, but not in $\mathcal{Q}^{(1)}_{\mathscr{D}_A}$ or $\mathcal{Q}^{(2)}_{\mathscr{D}_A}$. □

E. General USt codes

A similar procedure can be carried over for a general USt code $((n, K\,2^k, d))$, with the only difference that the definitions of the groups $\mathscr{D}$ and the auxiliary codes $\mathcal{Q}_{0\mathscr{D}}$ [Eq. (39)] should also include the k generators of the originating stabilizer code $\mathcal{Q}_0$ [Sec. II D]. Overall, the complexity of error recovery for a generic USt code can be summarized by the following

Theorem 4 *Consider any t-error correcting USt code of length n and dimension $K\,2^k$, with the translation set of size K. Then this code can correct errors using $\binom{n}{t}+2t-1$ or fewer measurements, each of which has complexity $2K(n+1)(n-k-1)$ or less.*

F. Error correction beyond *t*

For additive quantum codes, the syndrome measurement locates all error equivalence classes, not only those with "coset leaders" of weight $s < d/2$. The same could be achieved with a series of clustered measurements, by first going over all clusters of weight $s = t$, then $s = t+1$, etc. This ensures that the first located error has the smallest weight. In contrast, such a procedure will likely fail for a non-additive code where the corrupted spaces $E_1(Q)$ and $E_2(Q)$ can partially overlap if either E_1 or E_2 is non-correctable. For instance, the measurement in Example 6 may destroy the coherent superposition if the actual error (e.g., Z_i, $i \neq 2$) was not on the second qubit.

Therefore, if no error was detected after $\binom{n}{t}-1$ measurements, we can continue searching for the higher-weight errors only after testing the remaining size-t index set. With a non-additive CWS code, generally we have to do a separate measurement for each additional correctable error of weight $s > t$.

VI. CONCLUSIONS

For generic CWS and USt codes, we constructed a *structured recovery algorithm* which uses a single non-Pauli measurement to check for groups of errors located on clusters of t qubits. Unfortunately, for a generic CWS code with large K and large distance, both the number of measurements and the corresponding complexity are exponentially large, in spite of the exponential acceleration already achieved by the combined measurement.

To be deployed, error-correction must be complemented with some fault-tolerant scheme for elementary

gates. It is an important open question whether a fault-tolerant version of our measurement circuits can be constructed for non-additive CWS codes. It is clear, however, that such a procedure would *not* help for any CWS code that needs an exponential number of gates for recovery. Therefore, the most important question is whether this design can be simplified further.

We first note that the group-based recovery [see Theorem 2] is likely as efficient as it can possibly be, illustrated by the example of additive codes in Sec. V C where this procedure is shown to be equivalent to syndrome-based recovery. Also, while it is possible that for fixed K the complexity estimate of Theorem 1 can be reduced in terms of n (e.g., by reusing ancillas with measured stabilizer values), we think that for a generic code the complexity is linear in K.

However, specific families of CWS codes might be represented as unions of just a few stabilizer codes which might be mutually equivalent as in Eq. (15), or non-equivalent [28]. The corresponding measurement complexity for error detection would then be dramatically reduced. Examples are given by the quantum codes derived from the classical non-linear Goethals and Preparata codes [23, 24].

Another possibility is that for particular codes, larger sets of correctable errors may form groups. Indeed, we saw that for an additive code $((n, 2^k, d))$, all error degeneracy classes form a large group of size 2^{n-k} which may include some errors of weight well beyond t. Such a group also exists for a CWS code which is a subcode of an additive code. There could be interesting families of non-additive CWS codes which admit groups of correctable errors of size beyond 2^{2t}. For such a code, the number of measurements required for recovery could be additionally reduced.

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Appendix A: Orthogonality of corrupted spaces

As discussed in Sec. II B, for a general non-additive quantum code $\mathcal{Q}$ and two linearly independent correctable errors, the corrupted spaces $E_1(\mathcal{Q})$ and $E_2(\mathcal{Q})$ may be neither identical nor orthogonal [17]. However, for CWS and USt codes it is almost self-evident that when $E_1(\mathcal{Q})$ and $E_2(\mathcal{Q})$ do not coinside, they are mutually orthogonal. This orthogonality is inherited from the originating stabilizer code $\mathcal{Q}_0$. In particular, in some previous publications (e.g., Ref. [20]) orthogonality is implied in the discussion of degenerate errors for CWS codes. However, to our knowledge, it was never explicitly discussed for CWS or USt codes. Since our recovery algorithms for CWS and USt codes rely heavily on this orthogonality, we give here an explicit proof.

First, consider a stabilizer code $\mathcal{Q}_0$. For any Pauli operator $E \equiv E_1^\dagger E_2$, there are three possibilities: (**i**) E is proportional to a member of the stabilizer group, $E \equiv \gamma S$, where $S \in \mathscr{S}_0$ and $\gamma = i^m$, $m = 0, \ldots, 3$, (**ii**) E is in the code normalizer $\mathscr{N}_0$ but is linearly independent of any member of the stabilizer group, and (**iii**) E is outside of the normalizer, $E \notin \mathscr{N}_0$.

Case (**i**) implies that the space $E(\mathcal{Q}_0)$ is identical to the code $\mathcal{Q}_0$; the errors E_1 and E_2 are mutually degenerate. Indeed, for any basis vector $|i\rangle$, the action of the error $E\,|i\rangle = \gamma S\,|i\rangle = \gamma\,|i\rangle$ just introduces a common phase γ; any vector $|\psi\rangle \in \mathcal{Q}_0$ is mapped to $\gamma\,|\psi\rangle$ and hence no recovery is needed.

In case (**ii**) the operator E also maps $\mathcal{Q}_0$ to itself, but no longer identically. Therefore, at least one of the two errors E_1, E_2 is not correctable. Indeed, in this case we can decompose E (see Sec. II C) as the product of an element $S \in \mathscr{S}_0$ in the stabilizer and logical operators, i.e., $E \equiv i^m S \overline{X}^{\mathbf{a}} \overline{Z}^{\mathbf{b}}$, where $m = 0, \ldots, 3$ determines the overall phase. While $S \in \mathscr{S}_0$ acts trivially on the code, the logical operator specified by the binary-vectors $\mathbf{a}$, $\mathbf{b}$ is non-trivial, $\mathrm{wgt}(\mathbf{a}) + \mathrm{wgt}(\mathbf{b}) \neq 0$. Using the explicit basis (9), it is easy to check that the error-correction condition (5) is *not* satisfied for the operators E_1, E_2.

Finally, in case (**iii**) the spaces $E(\mathcal{Q}_0)$ and $\mathcal{Q}_0$ are mutually orthogonal. Indeed, since E is outside of the code normalizer $\mathscr{N}_0$, there is an element of the stabilizer group $S \in \mathscr{S}_0$ that does not commute with E. Therefore, for any two states in the code, $|\varphi\rangle, |\psi\rangle \in \mathcal{Q}_0$, we can write

$$\langle\varphi|\,E\,|\psi\rangle = \langle\varphi|\,ES\,|\psi\rangle = -\langle\varphi|\,SE\,|\psi\rangle = -\langle\varphi|\,E\,|\psi\rangle\,, \tag{A1}$$

which gives $\langle\varphi|\,E\,|\psi\rangle = 0$, and the spaces $\mathcal{Q}_0$ and $E(\mathcal{Q}_0)$ [also, $E_1(\mathcal{Q}_0)$ and $E_2(\mathcal{Q}_0)$] are mutually orthogonal.

Now, consider the same three cases for a USt code (15) derived from $\mathcal{Q}_0$. In case (**i**) the code is mapped to itself, $E(\mathcal{Q}) = \mathcal{Q}$. The operator E acts trivially on the code (and the errors E_1, E_2 are mutually degenerate) if E either commutes (A2) or anti-commutes (A3) with the entire set of translations generating the code:

$$(Et_j = t_j E,\ j = 1, \ldots, K) \tag{A2}$$

$$\text{or}\ \ (Et_j = -t_j E,\ j = 1, \ldots, K)\,. \tag{A3}$$

If neither of these conditions is satisfied, the error-correction condition (5) is violated. This is easily checked using the basis $|j, i\rangle \equiv t_j\,|i\rangle$.

Similarly, in case (**ii**), the code is mapped to itself, $E(\mathcal{Q}) = \mathcal{Q}$, but the error-correction condition (5) cannot be satisfied.

Finally, in case (**iii**), the space $E(\mathcal{Q})$ is either orthogonal to $\mathcal{Q}$, or the error correction condition is not satisfied. The latter is true if E is proportional to an element in

one of the cosets $t_j^\dagger t_{j'}\mathscr{S}_0$, where $j \neq j'$, $1 \leq j, j' \leq K$. Then the inner product $\langle j,i|\,E\,|j',i\rangle \neq 0$, $i = 1,\ldots,k$, which contradicts the error-correction condition (5). In the other case, namely, when E is linearly independent of any operator of the form $t_j^\dagger t_{j'}S$, $j,j' = 1,\ldots,K$, $S \in \mathscr{S}_0$, E must be a member of a different coset $t_\alpha\mathscr{S}_0$ of the stabilizer $\mathscr{S}_0$ of the code $\mathcal{Q}_0$ in $\mathscr{P}_n$. This implies orthogonality:

$$\langle j,i|\,E\,|j',i'\rangle \equiv \langle i|\,t_j^\dagger\,E\,t_{j'}\,|i'\rangle = i^m\,\langle i|\,t_j^\dagger t_{j'}\,t_\alpha\,|i'\rangle = 0,$$

where $m = 0,\ldots,3$ accounts for a possible phase factor.

Overall, as long as the error correction condition (5) is valid for a USt code $\mathcal{Q}$ and the Pauli operators E_1, E_2, the spaces $E_1(\mathcal{Q})$ and $E_2(\mathcal{Q})$ either coincide, or are orthogonal. Since CWS codes can be regarded as USt codes originating from a one-dimensional stabilizer code $\mathcal{Q}_0$, [Sec. II F], the same is also true for any CWS code.

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